

Quasi-Metric Spaces And Information Systems *

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SUMMARY

The relationship between information sources is defined in terms of a quasi-metric space. This property is used to quantify certain functional patterns in such man made information systems as libraries. Moreover, it is suggested that this spatial characteristic of information sources may be useful in explaining certain functional properties of the brain.

INTRODUCTION

Communication can be viewed as a process involving one or more objects, be they living organisms or devices, resulting in the conveying of information from one object to another. The first object is called a source and the second a destination. Weaver has defined three broad problem areas relating to the general problem of communication. These relate to 1) the accuracy with which the symbols of communication are transmitted (Engineering Problem), 2) the precision with which the transmitted symbols convey the desired information (Language Problem), and 3) the effectiveness with which the received information affects conduct in the desired way (Relevance Problem)¹. Although these three problem areas are interdependent, the engineering and language levels are essentially mechanisms for conveying relevant information from the source to the destination. Hence, the problem of relevance seems to be crucial to the general problem of communication. The celebrated Shannon theory, as is well known, does not directly deal with the relevance problem, nor does there seem to exist any other theory which addresses this problem. In this paper we shall define a general mathematical structure of an information source, based on the notion of relevance. This structure will then be used to describe certain organizational patterns in man made information systems such as a library and biological information systems such as the brain.

THE MATHEMATICAL STRUCTURE

A communication process can, in general, be characterized in terms of an element d , a set S and a relation R between the element d and the members of the set S . The element d stands for the destination, the set S for the information source, a collection of objects conveying information to d , and the relation R denotes the relevance relation between the information conveyed by the members of S , and d . Thus, associated with the set S is an ill defined and chaotic set \bar{S} together with a mapping $s \mapsto \bar{s}$ from S into \bar{S} . We interpret \bar{s} as the informational content of the information conveying object s . We do not assume that the mapping is one-to-one. Assume that relevance is measurable in terms of some value on the closed interval $[0,1]$. Let $M_d(s)$ denote the relevance measure of the information s conveyed by the object $s \in S$ relative to the destination d . Assume that associated with every destination d is a critical measure $J(d)$. Thus, dRs if and only if $M_d(s) > J(d)$, i.e., the information conveyed by s will be relevant to d if and only if the relevance measure of s exceeds the critical measure relative to d . The critical measure is thus a threshold, being the dividing line between the barely relevant and the not quite relevant information conveyed by the source. Since the information conveyed by two given members of the source S need not be independent, i.e., the relevance of the information conveyed by one member may either be inhibited or enhanced by the relevance of the information conveyed by the other, a dependency relation between the members of S must be defined.

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DEFINITION 1: Let H_{ij} be the relevance measure of s_j given that the information conveyed by s_i is relevant to d ; $m_{ij} = 1$.

DEFINITION 2: Let n_{ij} be the relevance measure of s_j given that the information conveyed by s_i is not relevant to d ; $n_{ij} = 0$. Thus, m_{ij} denotes a conditional relevance measure which does not depend on d but only on s_i and s_j . On the other hand, n_{ij} depends on s_j and d alone since it denotes the relevance relationship between d and s_j if d were in a zero state, i.e., if no relevant information has been conveyed to d by any other member of S . Since n_{ij} is independent of i we shall drop that subscript for convenience.

THEOREM 1: If the information (sp) conveyed by a sequence of elements $X = (s_0, s_1, \dots, s_k)$, $X \subset S$, is relevant to d , then $n_{ij} > g(d)$ for every s_j .

PROOF: Consider the arbitrary sequence $X = (s_0, s_1, \dots, s_k)$. The relevance measure $M_d(s_j)$, $j = 0, 1, \dots, k$ can be defined by recursion as follows:

$$M_d(s_0) = n_0 \\ M_d(s_j) = m_{H_j} M_d(s_{j-1}) + n_j(1 - M_d(s_{j-1})) \\ = n_j + \sum_{i=1}^j m_{H_i} M_d(s_{i-1}), j = 1, 2, \dots, k, \\ \text{where } X_{H_j} = m_{H_j} - n_j.$$

If $M_d(s_0) > \{(d)\}$ then $n_0 > g(d)$ and

$$m_{i,0} = n_0 > g(d).$$

If $M_d(s_j) > \{(d)\}$, $j = 1, 2, \dots, k$,

$$\text{then } X_{j-1} > (Kd) - n_j / M_d(s_{j-1}).$$

Suppose that $m_{j-1} < g(d)$. Then

$$(Kd) - n_j > M_d(s_{j-1}) > i(d) - n_j,$$

whence $M_d(s_j) > 1$, which is absurd.

Thus, the information conveyed by a sequence of objects belonging to the information source S will be relevant at the destination d , only if the conditional relevance measure exceeds the threshold for every object in the sequence. Since m is independent of any specific destination d , theorem 1 states a necessary condition for relevance given any threshold, hence any destination. It is thus possible to define relationships between the members of an information source S independent of the destination.

DEFINITION 3: Given information conveying objects s_i and s_j in S such that $m_{ij} > f$, $0 < f < 1$, then s_i and s_j are said to converse relative to f : conversance is denoted by $s_i C_s s_j$.

DEFINITION 4: A sequence of elements s_i to s_j in S such that $s_i C_s s_{i+1} \dots C_s s_j$ is

called a communication chain from s_i to s_j :

communication is denoted by $s_i C_s s_j$.

Clearly for the information conveyed by each member of a set of information conveying objects in the source S to be relevant to some destination d , it is necessary that there exist a communication chain of these members relative to the threshold associated with d .

DEFINITION 5: If $s_i C_s s_j$ and $s_j C_s s_i$, then s_i and s_j are said to intercommunicate: intercommunication is denoted by $s_i C_s s_j$.

We note that conversance is reflexive but neither symmetric nor transitive; that communication is both reflexive and transitive but not symmetric; and that intercommunication is reflexive, symmetric and transitive. Hence, the relation of communication constitutes a pre-order on the source S whereas intercommunication establishes an equivalence relation on S . Thus, for a given threshold f , the relation of intercommunication partitions the source S into disjoint intercommunication classes having the property that all members and only members of that class are in intercommunication with each other.

The length of a communication chain consisting of a single element is equal to 0; the length of a chain consisting of two elements is equal to 1; the length of a chain of three elements is equal to two; in general, the length of a communication chain of k elements is equal to $k-1$.

THEOREM 2: The length of the smallest communication chain between every two elements of an information source S constitutes a quasi-metric defined on S .

PROOF: Let q denote the smallest communication chain connecting s_i and s_j , arbitrary elements of S . Then

$$1) q(s_i, s_j) = 0 \text{ if and only if } s_i = s_j$$

$$2) q(s_i, s_j) + q(s_j, s_k) > q(s_i, s_k)$$

If there is no communication chain between s_i and s_j , then $q(s_i, s_j) = \infty$.

Thus, in terms of its informational properties, an information source S can be described as a quasi-metric space (S, q, f) for each value of f . The space is not metric since, in general,

$$q(s_i, s_j) \neq q(s_j, s_i) \text{ for arbitrary } s_i \text{ and } s_j \text{ in } S.$$

As f varies from 0 to 1, a family of quasi-metric spaces are defined on S . Each space belonging to this family is covered by a set of disjoint intercommunication classes since intercommunication is an equivalence relation on S . The value of f determines the fineness of the covering, i.e., increasing f leads to a finer covering whereas decreasing the threshold results in a cruder one. Thus, as the threshold approaches 1, the covering of S approaches a partitioning into intercommunication

classes each of which contains a single element. On the other hand, as the threshold approaches zero, the covering of S approaches a single intercommunication class, namely S itself. The lowering of ϵ constitutes a contraction mapping A of S onto itself, since if each element of S is mapped onto itself, at a lower value of ϵ , $q(A(s_i), A(s_j)) < q(s_i, s_j)$ for any two elements $s_i, s_j \in S$. Thus, as ϵ approaches 0 the quasi distances between elements approach 1, the intercommunication classes covering S are very large and the preorder manifested by the network of communication chains is destroyed since a large number of elements tend to intercommunicate at distance 1. In fact, the source S tends to approach a metric space of diameter 1, the diameter being $\sup\{q(s_i, s_j) : s_i, s_j \in S\}$.

In general, the family of quasi-metric spaces defines a sequence of partitions of S where each partition is a collection of sets having S for their union and such that each set is a subset of one of the preceding coverings.

Although no element belonging to a given intercommunication class I_j intercommunicates with any element belonging to any other intercommunication class, such elements may converse or communicate with each other. Hence, definitions 3, 4, and 5 can be extended as follows:

DEFINITION 6: An intercommunication class I_j is said to converse with an intercommunication class I_l if there exist elements $s_j \in I_j$ and $s_l \in I_l$ such that $s_j C s_l$.

DEFINITION 7: An intercommunication class I_l is said to communicate with an intercommunication class I_j if there exists a sequence of intercommunication classes such that $I_l C I_1 + I_1 C \dots C I_j$. If I_l communicates with I_j and I_j communicates with I_l then I_j and I_l are said to intercommunicate.

Thus there exists a network of communicating intercommunication classes which defines a preorder on the set of intercommunication classes covering S for each ϵ and a partition of the set into intercommunication classes of intercommunication classes. Moreover, a quasi distance between intercommunication classes can be defined on the basis of theorem 2.

A subset S_i of S is a subspace (S_i, q, ϵ) of the quasi-metric space (S, q, ϵ) . A subspace (S_i, q, ϵ) is called ϵ -complete if every communication chain containing elements in S_i belongs to S_i , i.e., if all elements in S of finite distance from the elements in S_i are also members of S_i . Thus, the space (S, q, ϵ) can be partitioned into ϵ -complete subspaces where each subspace represents a class of communicating

intercommunication classes which in turn can be covered by intercommunicating intercommunication classes.

Consequently, an arbitrary information source S is shown to be classifiable (intercommunication classes), orderable (communication chains) and metrizable (quasi-metric space). Moreover, this structure may exist at different levels of complexity (intercommunication classes of intercommunication classes, communication chains of intercommunication classes and distances between intercommunication classes). This structure defined on S introduces a similar structure in 2 . Hence we can speak of information being near or related. Because the mapping of S into 2 is not one-to-one, information may be replicated and diffuse throughout a region of S . The quasi distance between information conveying objects is, in a sense, a measure of this redundancy in the information source, those objects being close to each other conveying more redundant information than those that are further away. Because the distance is quasi, the order in which information is conveyed by the information conveying objects affects the amount of redundancy of this information.

Since communication is a dynamic process, the structure of the source is continuously undergoing transformation, mainly due to the following phenomena. Either new information enters the source or information is being lost due, for example, to the physical removal of certain of its information conveying objects. In general, these two phenomena occur simultaneously. Since every source can receive new information from other sources as well as convey it, each source can also be a destination relative to other sources and each destination can be a source relative to other destinations to which it may convey information. Thus, the structure of the set S , as defined above, represents the structure of both an arbitrary source and an arbitrary destination in the communication process.

Since every source S can be classified and ordered at various levels of complexity, it possesses the capability of both relating pieces of information which appear to be unrelated and of separating pieces of information which seem to be very closely related. That is, given elements s_i and s_j in S , there exists a ϵ such that $s_i C s_j$ and there exists a ϵ for which $q(s_i, s_j) = \infty$ and $q(s_j, s_j) = \infty$, i.e., such that there is no communication chain connecting s_j to s_j or vice versa. In the first case s_i and s_j , the information stored in s_i and s_j , is related because of the source's capability of generating a coarse enough covering to connect s_i with s_j , whereas in the latter case s_i and s_j are separated due to the source's capability of establishing a fine

enough covering of S to separate s_j from s_i . The range of values between O and I which ξ , can assume at any point in time governs the above capability. As new information enters S , there is a natural tendency for the range of values assumed by ξ to expand at the high end and contract at the low end. This follows from the fact that as the new information is assimilated ξ -complete subspaces of (S, q, ξ) which were disconnected for all values of ξ may become connected. Hence, it would take a higher range of values for ξ to maintain the source's informational structure. On the other hand, as information is lost there is a tendency for the range of values assumed by ξ to expand at the low end and contract at the high end. This is because it would take a lower value of ξ to reform classes which have been disconnected due to the loss of information and not as high a value of ξ to establish the finest coverings of S . Because of the redundancy in the information source, the information lost is partially regenerated by way of these new connections at lower thresholds.

Under normal conditions the range of threshold values of a given source which is continuously subject to the gain and loss of information, reaches and maintains a state of equilibrium. Thus the informational structure of an information source is maintained in a stable state. If, however, an abnormal event occurs, such as the massive loss of information due to the physical destruction of a large portion of S , then the ensuing lowering of the threshold range would partially regenerate the lost information. However, if the physical event is too severe and the resulting lowering of threshold range too great, all coverings of S will tend towards large intercommunication classes with small diameters leading to a breakdown of the ordering relation of communication among the information conveying objects in the source. Hence, the source or a portion of it could degenerate into a disorderly array of elements.

INFORMATION SYSTEMS

Information systems are mechanisms for carrying out communication processes. The oldest and still most widely used man-made information system is a library. In such a system the information source consists of a collection of documents which convey information to their users. In terms of the structure of an information source defined above, a library collection is represented by the set S , the subject matter or content of S is represented by S and the user by d . Then the family of ξ -complete quasi-metric spaces and their coverings by intercommunication classes represent a subject classification of the documents in S . The preorder defined in terms of communication

chains defined on S represents the relation between documents or classes of documents in the collection. Hence, the network of communication chains defined on S represents a subject catalogue. An intercommunication class would represent a subject literature, an intercommunication class of intercommunication classes the literature of a field and the ξ -complete quasi-metric spaces the literatures of disciplines. A query posed to the library system would result in an answer represented by a communication chain of documents. As new information enters the collection by way of acquired documents, the subject classification and catalogue should be adjusted to reflect a finer covering of the subject matter. On the other hand, if large portions of the collection were destroyed or deleted, the subject classification and catalogue would have to be adjusted to reflect a cruder covering of the subject matter. In the latter event, because of the redundancy inherent in the collection, the lost information would be partially regenerated from the documents in the informational neighborhoods of the deleted sets, namely from those documents containing information "near" to the deleted ones. Finally, if enough documents are destroyed, the collection would be reduced to a disorderly array devoid of structure.

The question now arises as to whether the structure of an arbitrary information source as developed above can be used to represent the functional structure of information in the nervous system. It should be noted that in discussing such a use of this mathematical structure, we will not be concerned with the biochemical and biophysical mechanisms of representing the information in physical form but are discussing the relationships in the functional organization of the brain which is induced by the energetics and the anatomical structures, just as in discussing the application of this structure to a library system we were not concerned with linguistic mechanisms or other physical properties of the documents in the collection but only in their informational relationships.

A property conferred by the mathematical representation of an information source is potential information regeneration. If all or a portion of an intercommunication class is removed, the information lost can be partially regenerated from the neighboring elements. This regenerative property is dependent upon the redundancy of the information throughout the family of quasi-metric spaces representing the informational structure of the nervous system, and the lowering of the threshold enough to cause development of the intercommunication relationship between elements previously related only by communication. If the informational structure is changed from its normal state by removal of an intercommunication class, we would expect from a threshold decrease to regenerate information by allowing the development of

intercommunication relationships to occur between replicates or partial replicates of the removed elements in the non-removed classes; the decrease in threshold also allows for new associations to occur among those classes and elements previously unrelated.

Assuming that normal ordered muscular activity is controlled by an ordered sequential relationship of informational elements (e.g. as represented by chains of neuron firings) in the nervous tissue which are stored and relayed to muscle tissue in some manner, a lowering of the threshold resulting from metabolic and/or structural change in the brain tissue or portion thereof, would result in a disorder among informational elements controlling motor activity. The muscular activity such as observed in epilepsy, seizures and spasms could therefore be considered symptomatic of disorder of the informational elements. Thus, a certain degree of order among the informational elements of the brain must exist for normal muscular activity but that disordered activity such as observed in epilepsy, results when there is a decrease in the range of thresholds involving the subspace representing the informational elements which controls muscular activity. Just below the normal range of thresholds in the mathematical structure, a state exists in which the information source has a greater degree of disorder than in the normal state but far less than if the threshold range were near zero. We infer from this that some type of abnormal human behavior should correspond to this state of disorder existing somewhere between the normal state and the state of disorder causing epilepsy. We suggest that this intermediate state is the aura or *deja vu* observed in the epileptic. This state is a result of a decreasing threshold which allows in a non specific manner previously ordered communication chains of elements to make new connections with other communication chains creating information in the form of memories. Unless the continuing decrease in the threshold range is inhibited in some manner, disordered motor activity will result. Using the mathematical structure as a heuristic device, a prediction was made that clinical observations should exist which relate brain damage, *deja vu*, and seizures. Maitland Baldwin² has pointed out that memories are never elicited unless the patient (undergoing electrical stimulation during surgical treatment of epilepsy) has had a past history of epilepsy involving damaged tissue near the part of the brain where stimulation elicits the experiential recall. Baldwin has inferred from this that the phenomenon is dependent upon lowering of the normal threshold of the memory recall process, resulting from current injected into the tissue by previous epileptic attacks.

It is estimated that 30-50 per cent of patients suffering brain damage also suffered observed seizures. Because of this high correlation of

epileptic seizures with brain damage, it seems reasonable to hypothesize that in the human nervous system a non-specific process operates to reestablish informational integrity so as to compensate for damage and the consequent loss of stored information. It would therefore follow that epilepsy is symptomatic of an overreaction of this compensating process spilling into areas of the brain responsible for muscular control and causing a lowering of the threshold range in areas not specifically damaged. The metabolic reaction of nervous tissue to any change is to cause a lowering of the threshold range and since the elements of information may exist in other non-damaged areas, to replace those elements lost in the affected portion. A process such as this also bypasses the necessity of depending upon tissue regeneration to re-establish informational integrity; thus giving a reasonable basis for the observation that nervous tissue regenerates at a very low and slow level, if at all. In addition, the lowering of the threshold range in small limited areas in the brain would provide for the new association of informational elements and thus a means of specific adaptation of the individual to new or changed environments. If the hypothesis is valid that a mechanism defined by a quasi-metric space is operative in nervous tissue both as a defense against the loss of vital information and for use in adaptation to new environments, it becomes apparent that in evolutionary development, only those organic compounds and cellular structures could survive and reproduce which possessed the ability to form and maintain a family of quasi-metric spaces for information storage, processing and control.

REFERENCES

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RESUMO

O relacionamento entre fontes de informação é definido em termos de um espaço quase-métrico. Esta propriedade é utilizada para quantificar certos padrões funcionais em sistemas de informação criados pelo homem, tais como a biblioteca. Além do mais, é sugerido que esta característica espacial das fontes de informação pode ser útil para explicar certas propriedades funcionais do cérebro.